Extended Decision Procedure for a Fragment of HL with Binders*

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Abstract

This work is a brief presentation of an extension of the proof procedure for a decidable fragment of hybrid logic presented in a previous paper. It shows how to extend such a calculus to multi-modal logic enriched with transitivity and relation inclusion assertions.

A further result reported in this work is that the logic extending the considered fragment with the addition of graded modalities has an undecidable satisfiability problem, unless further syntactical restrictions are placed on their use.

1 Introduction

This brief presentation is not self-contained, but can rather be considered as an addendum to [3]. The hybrid languages considered here rely on a multi-modal base, allowing for modelling structures with different accessibility relations. The basic modalities $\Diamond$ and $\Box$ (and their converses, if present) are indexed by relation symbols. The possibility of declaring an accessibility relation to be transitive and/or included in another one is also considered.

In this work, basic hybrid logic (with nominals only, beyond the modal operators $\Diamond$ and $\Box$) will be denoted by HL, and basic multi-modal hybrid logic by HL_m. Logics extending HL or HL_m with operators $O_1, \ldots, O_n$ (and their duals) are denoted by $\mathsf{HL}(O_1, \ldots, O_n)$ and $\mathsf{HL}_m(O_1, \ldots, O_n)$, respectively. Multimodal languages including transitivity assertions and/or relation hierarchies are denoted in the same way, just including Trans (for transitivity) and/or $\sqsubseteq$ (for relation inclusion) among $O_1, \ldots, O_n$.

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The notations $\mathsf{HL}(O_1,\ldots,O_n)\setminus\Box{\downarrow}\Box$ and $\mathsf{HL}_m(O_1,\ldots,O_n)\setminus\Box{\downarrow}\Box$ are used to denote the sets of formulae of $\mathsf{HL}(O_1,\ldots,O_n)$ and, respectively, $\mathsf{HL}_m(O_1,\ldots,O_n)$, whose negation normal form (NNF) do not contain any binder that is both in the scope of and has in its scope a universal modality (i.e. either $\Box_R$ or the universal global modality A, if present). Analogously, $\mathsf{HL}(O_1,\ldots,O_n)\setminus{\downarrow}\Box$ and $\mathsf{HL}_m(O_1,\ldots,O_n)\setminus{\downarrow}\Box$ are the fragments of $\mathsf{HL}(O_1,\ldots,O_n)$ and, respectively, $\mathsf{HL}_m(O_1,\ldots,O_n)$, consisting of formulae whose NNF have no universal operator occurring in the scope of a binder.

The fact that $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-)\setminus\Box{\downarrow}\Box$ has a decidable satisfiability problem can be proved by the same reasoning used in [14] for $\mathsf{HL}(@,\downarrow,\mathsf{E},\Diamond^-)\setminus\Box{\downarrow}\Box$. The separate addition of either transitive relations or relation hierarchies to the considered decidable fragment of multi-modal hybrid logic can easily be shown to stay decidable, by resorting to results already proved in the literature. However, such results do not directly allow for concluding whether the logic including both features is still decidable. The existence of the terminating, sound and complete calculus for the considered logic, presented in this work, proves that the addition of transitive relations and relation hierarchies to such an expressive decidable fragment of hybrid logic does not endanger decidability.

The first paper presenting a decision procedure for a syntactically restricted fragment of hybrid logic is [2], followed by [3], where a tableau based satisfiability decision procedure for $\mathsf{HL}(@,\downarrow,\mathsf{E},\Diamond^-)\setminus\Box{\downarrow}\Box$ is defined. Such a procedure is extended to multi-modal hybrid logic $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box{\downarrow}\Box$ in [4]. This work presents the main guidelines of the latter and a new result concerning graded modalities. The procedure is based on a tableau calculus which terminates and is sound and complete for formulae in the fragment $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus{\downarrow}\Box$. A preprocessing step along the lines of [14] turns the calculus into a satisfiability decision procedure for the fragment $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box{\downarrow}\Box$. An extended, self-contained version of this work, including full proofs, is [5].

This work shows also that graded modalities cannot, in general, be added to $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box{\downarrow}\Box$ without endangering decidability: the satisfiability problem for hybrid logic with either the satisfaction operator or converse modalities, functionality restrictions and binders, without the critical pattern $\Box{\downarrow}\Box$, is undecidable. However, decidability can be preserved by placing some strong additional syntactical restrictions on the occurrences of graded modalities.

This section concludes with the definition of the syntax and semantics of multi-modal hybrid logic with transitive relations and inclusion assertions.

Well-formed expressions of $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)$ are partitioned into two categories: *formulae* (for which the metasymbols F,G,H are used) and *assertions*. The language is based on a set PROP of propositional letters, a set NOM of nominals, an infinite set VAR of state variables, and a set REL of relation symbols (all such sets being mutually disjoint). When using a meta-symbol r for a relation symbol in REL, the corresponding uppercase letter, R, will denote a *relation*, i.e. either the relation denoted by r itself (a *forward relation*) or its converse, denoted by r^- (a *backward relation*). A backward relation r^- denotes the set of pairs of states $\langle w, w'\rangle$ such that $\langle w', w\rangle$ is in the relation denoted by r.

Formulae are defined by the following grammar:

$$F := p \mid u \mid \neg F \mid F \wedge F \mid F \vee F \mid \Diamond_R F \mid \Box_R F \mid \mathsf{E}F \mid \mathsf{A}F \mid u{:}F \mid {\downarrow} x.F$$

where $p \in \mathsf{PROP}$, $u \in \mathsf{NOM} \cup \mathsf{VAR}$, $x \in \mathsf{VAR}$ and R is either a forward or backward relation. The metavariables a, b, c, d are used for nominals, x, y, z for state variables and r, s, t for relation symbols (every metavariable possibly decorated by subscripts). Although the meta-symbol @ is used to denote the satisfaction operator, in this work the notation $t{:}F$ is used (for $t \in \mathsf{NOM} \cup \mathsf{VAR}$) rather than $@_t F$. The modal operators A and $\Box_R$ are called universal modalities.

Assertions are either *transitivity assertions,* of the form $\mathsf{Trans}(r)$, for $r \in \mathsf{REL}$, or *inclusion assertions,* of either form $r \sqsubseteq s$ or $r^- \sqsubseteq s$, for $r, s \in \mathsf{REL}$. Note that backward relations are allowed only on the left of the $\sqsubseteq$ symbol. This is only a syntactical restriction, and expressions of the form $R \sqsubseteq S$ are used as abbreviations of their semantically equivalent assertions: $r^- \sqsubseteq s^-$ stands for $r \sqsubseteq s$, and $r \sqsubseteq s^-$ for $r^- \sqsubseteq s$.

An *interpretation* $\mathcal{M}$ of an $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq)$ language is a tuple $\langle W, \rho, N, I\rangle$ where W is a non-empty set (whose elements are the *states* of the interpretation), ρ is a function mapping every $r \in \mathsf{REL}$ to a binary relation on W ($\rho(r) \subseteq W \times W$), N is a function $\mathsf{NOM} \to W$ and I a function $W \to 2^{\mathsf{PROP}}$. The following abbreviation will be used:

$$wRw' \;\equiv_{\text{def}} \begin{cases} \langle w, w'\rangle \in \rho(r) & \text{if } R = r \text{ is a forward relation} \\ \langle w', w\rangle \in \rho(r) & \text{if } R = r^- \text{ is a backward relation} \end{cases}$$

If $\mathcal{M} = \langle W, \rho, N, I\rangle$ is an interpretation, $w \in W$, σ is a variable assignment for $\mathcal{M}$ (i.e. a function $\mathsf{VAR} \to W$) and F is a formula, the relation $\mathcal{M}_w, \sigma \models F$ is defined like in [3], just replacing the clauses for the uni-modal operators with the following ones:

- $\mathcal{M}_w, \sigma \models \Box_R F$ if for every w' such that wRw', $\mathcal{M}_{w'}, \sigma \models F$.
- $\mathcal{M}_w, \sigma \models \Diamond_R F$ if there exists w' such that wRw' and $\mathcal{M}_{w'}, \sigma \models F$.

The definition of other basic notions, such as formula satisfiability, logical equivalence and negation normal form (NNF) are standard.

If $\mathcal{A}$ is a set of assertions, an interpretation $\langle W, \rho, N, I\rangle$ is a model of $\mathcal{A}$ if:

1. for all $r \in \mathsf{REL}$ such that $\mathsf{Trans}(r) \in \mathcal{A}$, $\rho(r)$ is a transitive relation;
2. for all $r, s \in \mathsf{REL}$, if $r \sqsubseteq s \in \mathcal{A}$, then $\rho(r) \subseteq \rho(s)$;
3. for all $r, s \in \mathsf{REL}$ and all $w, w' \in W$, if $r^- \sqsubseteq s \in \mathcal{A}$ and $\langle w, w'\rangle \in \rho(r)$, then $\langle w', w\rangle \in \rho(s)$.

If $\mathcal{A}$ is a set of assertions and F a formula, then $\mathcal{A} \cup \{F\}$ is satisfiable if there exist a model $\mathcal{M}$ of $\mathcal{A}$, a state w of $\mathcal{M}$ and a variable assigment σ for $\mathcal{M}$ such that $\mathcal{M}_w, \sigma \models F$.

2 The Proof Procedure

Let F be a formula in NNF belonging to the fragment $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$ and $\mathcal{A}$ a set of assertions. In order to test $\{F\} \cup \mathcal{A}$ for satisfiability by means of the tableau calculus presented below, F is first preprocessed and translated into an equisatisfiable formula in the fragment $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\!\Box$. The translation is the multi-modal analogous of the (polynomial) satisfiability preserving translation given in [14] for $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$.

The rest of this section describes how to integrate transitivity and inclusion assertions into the system described in [3], obtaining a tableau calculus for $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq)$.

In the presence of multi-modalities, *relational formulae* are statements of the form $a{:}\Diamond_r b$, where a and b are nominals and r is a forward relation. Expressions of the form $a \Rightarrow_R b$ will be used as abbreviations for relational formulae:

$$a \Rightarrow_R b \qquad \equiv_{\text{def}} \quad \begin{cases} a{:}\Diamond_r b & \text{if } R = r \\ b{:}\Diamond_r a & \text{if } R = r^- \end{cases}$$

By convention, an expression of the form $\mathsf{Trans}(R)$, where R is a meta-symbol standing for either a forward or backward relation, will stand for $\mathsf{Trans}(r)$, where $r \in \mathsf{REL}$ is the relation symbol in R.

Let F be a ground hybrid formula in NNF and $\mathcal{A}$ a set of assertions. A tableau for $\{F\} \cup \mathcal{A}$ is initialized with a single branch, constituted by the node $(n_0)\, a_0{:}F$, where a_0 is a new nominal, followed by nodes labelled by the assertions in $\mathcal{A}$ and then expanded according to the *Assertion rules* of Table 1 (note that Rel actually stands for four rules, according to the relation signs). These rules complete the inclusion assertions in $\mathcal{A}$ by the reflexive and transitive closure of $\sqsubseteq$. The *initial formula* of the tableau is $a_0{:}F$.

$$\frac{}{r \sqsubseteq r}\ \mathsf{Rel}_0 \qquad\qquad \frac{R \sqsubseteq S \quad S \sqsubseteq T}{R \sqsubseteq T}\ \mathsf{Rel}$$

Table 1: Assertion rules

The other expansion rules introduced to treat assertions are similar to the analogous ones presented in [8, 9, 10, 11]. They are shown in Table 2, along with the reformulation of the $\Box$ and $\Diamond$ rules for multi-modalities (the $\Diamond$ rule is applicable only if the label of its premiss is not a relational formula). Note that using uppercase meta-symbols for relations is just a syntactical expedient allowing different rules to be presented in a single schema.

The premiss n of not only the $\Box$ rule, but also the Trans rule is called the *major premiss*, and m the *minor premiss* of the rule. In an application of the Link rule, n is its *logical premiss*.

Universal and blockable formulae and nodes are defined like in [3]. Convening that the *top node* of a branch is its first node, also the notions of top formula and top nominal stay the same.

$$\frac{(n)\, a\colon \Box_R F \quad (m)\, a \Rightarrow_R b}{(k)\, b\colon F}\ (\Box) \qquad \frac{(n)\, a\colon \Diamond_R F}{\begin{array}{c}(m_0)\, a \Rightarrow_R b\\ (m_1)\, b\colon F\end{array}}\ (\Diamond)$$

$$\frac{(n)\, a \Rightarrow_R b \quad (i)\, R \sqsubseteq S}{(m)\, a \Rightarrow_S b}\ (\mathsf{Link})$$

$$\frac{(n)\, a\colon \Box_S F \quad (m)\, a \Rightarrow_R b \quad (t)\, \mathsf{Trans}(R) \quad (i) R \sqsubseteq S}{(k)\, b\colon \Box_R F}\ (\mathsf{Trans})$$

Table 2: The new expansion rules

The Trans rule deals with transitive relations and can be seen as a reformulation (in the presence of inclusion assertions) of the $\Box$ rule for transitive modal logics (a particular case of this rule is when $R = S$). The formulation of the Trans rule of Table 2 is very close to the corresponding one used in description logics, where in fact "roles" include both *role names* (corresponding to relation symbols) and the inverse of role names, and inverse roles may also occur in role inclusion axioms. The abbreviation $a \Rightarrow_R b$, however, does not have exactly the same meaning as the corresponding premiss used in the rule treating transitivity in description logics [8, 9] (a similar approach is adopted in [10]), consisting of the meta-notion "b is an R-neighbour of a". There are two main differences between the two approaches. First of all, the semantical notion of accessibility between two states is here given a "canonical representation" in the object language (a choice already made in [2, 3]): the fact that a state a is r-related to b is represented by the *relational formula* $a\colon \Diamond_r b$. Though semantically equivalent to $b\colon \Diamond_{r^-} a$, the latter is not a relational formula, i.e. it is not the canonical representation of an r-relation. Moreover, in the present work, the notation $a \Rightarrow_R b$ is only an abbreviation for a relational formula, which does not take subrelations into account: it may be the case that $a \Rightarrow_S b$ belongs to a given branch for some $S \sqsubseteq R$, and yet $a \Rightarrow_R b$ does not. The fact that, in the present work, no meta-notion is used to represent "R-neighbours" is responsible for the presence of the Link rules, which have no counterpart in [8, 9, 10].

An approach that, in the above respect, shares some similarities with the present one is represented by [12], where a tableau calculus for $\mathcal{SHOI}$ is proposed. In that work, relations between individuals are explicitly represented by use of expressions similar to relational formulae, and, in fact, the description logic counterpart of the Link rule is included in the calculus. The calculus however enjoys only a form of weak termination.

In the presence of the Trans rule, the important *strong subformula property* of the calculus must be reformulated: provided that the initial formula is in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\Box$, every universal formula occurring in a tableau branch is obtained from a subformula of the top formula of the branch by possibly replacing operators $\Box_R$ with $\Box_S$, for some relation S in the language of the initial tableau

Direct blocking is defined like in [3]. Also the indirect blocking relation stays the same, but for the fact that, in order to cope with the new rules, the *offspring relation* $\prec_{\mathcal{B}}$ must be extended accordingly:

1. all the nodes of the initial tableau are root nodes;
2. if a node k has been added to the branch by an application of the Trans rule, then k is a sibling of the minor premiss m of the inference (i.e., if m is a root node, then k is a root node too; otherwise, if $k' \prec_{\mathcal{B}} m$, then $k' \prec_{\mathcal{B}} k$);
3. if if a node k has been added to the branch by an application of the Link rule, then k is a sibling of the logical premiss of the inference.

Note that the Trans rule is treated like the universal rules, since one of its premisses is a universal node. Restriction **R3** (Definition 5 in [3]) also applies to the Link and Trans rules: a phantom node cannot be used as the logical premiss of an application of the Link rule, nor can it be used as the minor premiss of the Trans rule.

The guidelines of the termination and completeness proofs given in [3] can be essentially reused for this extended calculus, except for the model contruction in the completeness proof, which needs a different approach. Details can be found in [5].

It is worth pointing out that, according to the termination proof, the worst-case complexity of the calculus presented in this work has the same order of magnitude of the calculus in [3], since the number of nodes in a single branch is bounded by a doubly exponential function of the size of the input problem. As a consequence, the tableau calculus presented in this work shows that the satisfiability problem for $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box{\downarrow}\Box$ is in 2-NExpTime. It is reasonable to hypothesize that the problem complexity is actually lower and consequently that, like many other tableau based algorithms, the decision procedure defined in this paper is not worst-case optimal.

This section concludes with an example illustrating the calculus in action.[1] Figure 1 represents a complete and open tableau branch $\mathcal{B}$ for the assertion $\mathsf{Trans}(r)$ and the formula

$$F = \Diamond_r \top \wedge \mathsf{A}\Box_{r^-} p \wedge \Box_r G \quad \text{where} \quad G = {\downarrow}x.\Diamond_r{\downarrow}y.x{:}\,\Diamond_r \neg y$$

When r is transitive, F holds at a state w of an interpretation $\mathcal{M}$ if w has at least one r-successor, all its r-descendants have at least two different r-successors and every state of the model with at least one r-successor satisfies p.

In Figure 1, the notations $n \leadsto^{\mathcal{R}} m$ or $(n_1, \ldots, n_k) \leadsto^{\mathcal{R}} m$ mean that the addition of node m is due to the application of rule $\mathcal{R}$ to node n (or nodes $n_1, \ldots, n_k$). If $\mathcal{R} = \mathsf{A}$, then also the minor premiss is indicated: $(n, m) \leadsto^{\mathsf{A}} k$

[1] The second example in [4] (Figure 1) is incorrect. An *Errata corrige* is available at the author's web page.

means that the A rule is applied to n with minor premiss m, producing k. In the comments that follow, the notation $\mathcal{B}_n$ is used to denote the branch segment up to node n included and the $n \prec_{\mathcal{B}} \{m_1, \dots, m_k\}$, used to illustrate the offspring relation, abbreviates $n \prec_{\mathcal{B}} m_1$ and $\dots n \prec_{\mathcal{B}} m_k$.

0)	$a_0\colon F$	
1)	$\mathsf{Trans}(r)$	
2)	$r \sqsubseteq r$	
3)	$a_0\colon (\Diamond_r \top \wedge \mathsf{A}\Box_{r^-} p)$	$0 \leadsto^{\wedge} 3$
4)	$a_0\colon \Box_r G$	$0 \leadsto^{\wedge} 4$
5)	$a_0\colon \Diamond_r \top$	$3 \leadsto^{\wedge} 5$
6)	$a_0\colon \mathsf{A}\Box_{r^-} p$	$3 \leadsto^{\wedge} 6$
7)	$a_0\colon \Diamond_r a_1$	$5 \leadsto^{\Diamond} 7$
8)	$a_1\colon \top$	$5 \leadsto^{\Diamond} 8$
9)	$a_1\colon \Box_{r^-} p$	$(6,7) \leadsto^{\mathsf{A}} 9$
10)	$a_0\colon \Box_{r^-} p$	$(6,0) \leadsto^{\mathsf{A}} 10$
11)	$a_1\colon \Box_r G$	$(4,7,1,2) \leadsto^{\mathsf{Trans}} 11$
12)	$a_1\colon G$	$(4,7) \leadsto^{\Box} 12$
13)	$a_0\colon p$	$(9,7) \leadsto^{\Box} 13$
14)	$a_1\colon \Diamond_r {\downarrow} y.a_1\colon \Diamond_r \neg y$	$12 \leadsto^{\downarrow} 14$
15)	$a_1\colon \Diamond_r a_2$	$14 \leadsto^{\Diamond} 15$
16)	$a_2\colon {\downarrow} y.a_1\colon \Diamond_r \neg y$	$14 \leadsto^{\Diamond} 16$
17)	$a_2\colon \Box_r G$	$(11,15,1,2) \leadsto^{\mathsf{Trans}} 17$
18)	$a_2\colon G$	$(11,15) \leadsto^{\Box} 18$
19)	$a_2\colon a_1\colon \Diamond_r \neg a_2$	$16 \leadsto^{\downarrow} 19$
20)	$a_2\colon \Diamond_r {\downarrow} y.a_2\colon \Diamond_r \neg y$	$18 \leadsto^{\downarrow} 20$
21)	$a_1\colon \Diamond_r \neg a_2$	$19 \leadsto^{@} 21$
22)	$a_2\colon \Diamond_r a_3$	$20 \leadsto^{\Diamond} 22$
23)	$a_3\colon {\downarrow} y.a_2\colon \Diamond_r \neg y$	$20 \leadsto^{\Diamond} 23$
24)	$a_1\colon \Diamond_r a_4$	$21 \leadsto^{\Diamond} 24$
25)	$a_4\colon \neg a_2$	$21 \leadsto^{\Diamond} 25$
26)	$a_3\colon \Box_r G$	$(17,22,1,2) \leadsto^{\mathsf{Trans}} 26$
27)	$a_3\colon G$	$(17,22) \leadsto^{\Box} 27$
28)	$a_3\colon a_2\colon \Diamond_r \neg a_3$	$23 \leadsto^{\downarrow} 28$
29)	$a_4\colon \Box_r G$	$(11,24,1,2) \leadsto^{\mathsf{Trans}} 29$
30)	$a_4\colon G$	$(11,24) \leadsto^{\Box} 30$
31)	$a_2\colon \Diamond_r \neg a_3$	$28 \leadsto^{@} 31$
32)	$a_4\colon \Diamond_r {\downarrow} y.a_4\colon \Diamond_r \neg y$	$30 \leadsto^{\downarrow} 32$
33)	$a_2\colon \Diamond_r a_5$	$31 \leadsto^{\Diamond} 33$
34)	$a_5\colon \neg a_3$	$31 \leadsto^{\Diamond} 34$
35)	$a_4\colon \Box_{r^-} p$	$(6,24) \leadsto^{\mathsf{A}} 35$
36)	$a_3\colon \Box_{r^-} p$	$(6,22) \leadsto^{\mathsf{A}} 36$
37)	$a_2\colon \Box_{r^-} p$	$(6,15) \leadsto^{\mathsf{A}} 37$
38)	$a_1\colon p$	$(35,24) \leadsto^{\Box} 38$
39)	$a_2\colon p$	$(36,22) \leadsto^{\Box} 39$
40)	$a_4\colon \Diamond_r a_6$	$32 \leadsto^{\Diamond} 40$
41)	$a_6\colon {\downarrow} y.a_4\colon \Diamond_r \neg y$	$32 \leadsto^{\Diamond} 41$
42)	$a_6\colon \Box_{r^-} p$	$(6,40) \leadsto^{\mathsf{A}} 42$
43)	$a_4\colon p$	$(42,40) \leadsto^{\Box} 43$

Figure 1: A complete tableau branch for $\{\Diamond_r \top \wedge \mathsf{A}\Box_{r^-} p \wedge \Box_r G, \mathsf{Trans}(r)\}$, where $G = {\downarrow} x.\Diamond_r {\downarrow} y.x\colon \Diamond_r \neg y$.

In this example, the formulae to be taken into account to check compatibilities are p, $\Box_{r^-} p$ and $\Box_r G$.

The root nodes are 0–6 and 10, and the offspring relation is:

$5 \prec_{\mathcal{B}} \{7-9, 11-14\}$ $\quad$ $14 \prec_{\mathcal{B}} \{15-21, 37\}$
$20 \prec_{\mathcal{B}} \{22, 23, 26-28, 31, 36, 39\}$ $\quad$ $21 \prec_{\mathcal{B}} \{24, 25, 29, 30, 32, 35, 38\}$
$31 \prec_{\mathcal{B}} \{33, 34\}$ $\quad$ $32 \prec_{\mathcal{B}} \{40-43\}$

For instance, node 7 is the minor premiss of the application of the Trans rule producing 11, and the minor premiss of the application of the $\Box$ rule producing 12 and 13; therefore 7, 11, 12 and 13 are siblings. Moreover, 7 is also the first non-phantom node where a_1 occurs when the A rule is applied to produce node 9 focusing on a_1, therefore 7 is the minor premiss of the inference, thus one of 9's siblings.

As a further example, though node 22 is a phantom in the final branch, it is not a phantom in $\mathcal{B}_{35}$ (see below). The branch $\mathcal{B}_{35}$ is expanded by an application of the A rule focusing on a_3 and producing node 36. In this branch, 22 is the first non-phantom node where a_3 occurs, so it is the minor premiss of the A inference and 22 and 36 are siblings (in all branch segments from $\mathcal{B}_{36}$ onwards).

In the whole branch $\mathcal{B} = \mathcal{B}_{43}$, the nodes 20 and 32 are blocked by 14, because a_1 is compatible with both a_2 and a_4: the relevant formulae such nominals label in the final branch are p, $\Box_{r^-} p$ and $\Box_r G$.

The fact that 20 and 32 are blocked by 14 intuitively means that a_2 and a_4 behave "like" a_1, However, though a_2 and a_4 are compatible, the presence of node 25 does not allow to identify the states they denote in a model of this open branch.

Nodes 20 and 32 being directly blocked in $\mathcal{B}$, all their descendants $(22, 23, 26–28, 31, 33, 34, 36, 39–43)$ are phantom nodes in $\mathcal{B}$.

However, node 20 is blocked by 14 only in $\mathcal{B}_{37}$ (where a_1 and a_2 label $\Box_{r^-} p$ and $\Box_r G$) and from $\mathcal{B}_{39}$ onwards, when both (38) $a_1 \colon p$ and (39) $a_2 \colon p$ are added. In particular, 20 is not blocked in $\mathcal{B}_i$ for $i \leq 36$, therefore, it is expanded, and its descendants can also be expanded (or used as minor premisses) till node 39 is added to the branch.

Analogously, 32 is blocked by 14 in $\mathcal{B}_i$ only for $35 \leq i \leq 37$ and $i = 43$. Therefore, for instance, node 40 is not a phantom in $\mathcal{B}_{42}$, so that it can be used as the minor premiss of the application of the $\Box$ rule producing 43. Note also that in $\mathcal{B}_{38}$, where 20 is not blocked, a_2 and a_4 are compatible, therefore 20 blocks 32 in this branch segment (though 20 is not an ancestor of 32 w.r.t. the offspring relation).

In order for node 31 to be blocked by 21, a_1, a_2 and a_3 must be compatible. But when a_1 and a_2 are compatible, node 20 is blocked, and in such a case 31, that is one of 20's children, is a phantom. Therefore 31 is never directly blocked.

The branch is complete: no further expansion is possible without violating the restrictions on blocked nodes. In particular, in the whole branch:

- the A rule cannot focus on a_5, which only occurs in phantom nodes.
- Though nodes 36 and 42, obtained by applications of the A rule, are phantoms, this rule cannot focus again on a_3 and a_6, which only occur in phantom nodes.
- Though 26 and 27 are phantoms, the Trans and $\Box$ rules cannot use again 22 as a minor premiss, since it is a phantom too.
- Similarly, the other phantom nodes labelled by relational formulae cannot be used as minor premisses. For instance, 40 cannot be used as the minor premiss of an application of the $\Box$ rule, paired with 29.

3 Graded modalities

Graded modalities are here denoted by $\Diamond_r^n$ and $\Box_r^n$, where $n \in \mathbb{N}$. In the presence of converse modalities, also graded modalities indexed by backward relations are allowed, so their general forms are $\Diamond_R^n$ and $\Box_R^n$. Their semantics is the following:

- $\mathcal{M}_w, \sigma \models \Diamond_R^n F$ iff there are at least $n+1$ distinct states $w_1, ..., w_n$ such that wRw_i and $\mathcal{M}_{w_i}, \sigma \models F$.
- $\mathcal{M}_w, \sigma \models \Box_R^n F$ iff there are at most n distinct states $w_1, ..., w_n$ such that wRw_i and $\mathcal{M}_{w_i}, \sigma \not\models F$.

Obviously, $\Box_r^1$ can be used to express functionality of the relation $\rho(r)$, and analogously, $\Box_{r^-}^1$ can be used to express inverse functionality, Formulae of the form $\Box_R^1 \bot$ will be called *functionality restrictions*, and, if R is a forward relation, they are called *forward functionality restrictions*, otherwise *backward functionality restrictions*.

When considering the interplay between the binder and universal modalities in order to tackle decidability issues, universal graded modalities $\Box_R^n$ are to be included, along with $\Box_R$ and A, among the universal modalities (with the obvious consequence on the meaning of the patterns $\Box{\downarrow}\Box$ and ${\downarrow}\Box$).

If no further syntactical restrictions are placed on graded modalities, the satisfiability problem for $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \Diamond^n) \setminus \Box{\downarrow}\Box$ is undecidable. In order to establish this fact, only functionality restrictions are required.

In what follows, $\mathsf{HL}_m(@, \downarrow, \Diamond^1)$ denotes the hybrid multi-modal language with the satisfaction operator, the binder and forward functionality restrictions. Correspondingly, $\mathsf{HL}_m(@, \downarrow, \Diamond^1) \setminus \Box{\downarrow}\Box$ denotes the fragment of $\mathsf{HL}_m(@, \downarrow, \Diamond^1)$ consisting of formulae whose NNF do not contain any occurrence of the binder that is both in the scope and has in its scope a universal modality (i.e. either $\Box_r$ or $\Box_r^1$). The hybrid multi-modal language with binders, converse modalities and both forward and backward functionality restrictions will be denoted by $\mathsf{HL}_m(\downarrow, \Diamond^-, \Diamond^1)$, and its fragment consisting of formulae whose NNF do not contain any occurrence of the binder that is both in the scope and has in its scope a universal modality (i.e. either $\Box_R$ or $\Box_R^1$) is denoted by $\mathsf{HL}_m(\downarrow, \Diamond^-, \Diamond^1) \setminus \Box{\downarrow}\Box$.

Theorem 1. *The satisfiability problem for both* $\mathsf{HL}_m(@, \downarrow, \Diamond^1) \setminus \Box{\downarrow}\Box$ *and* $\mathsf{HL}_m(\downarrow, \Diamond^-, \Diamond^1) \setminus \Box{\downarrow}\Box$ *is undecidable.*

Proof. The proof is based on a modification of the encoding of the $\mathbb{N} \times \mathbb{N}$ tiling problem presented in [14] in order to prove that HL with binders is undecidable, even in the absence of the satisfaction operator.

The $\mathbb{N} \times \mathbb{N}$ tiling problem can be reduced to the satisfiability problem for $\mathsf{HL}_m(@, \downarrow, \Diamond^1) \setminus \Box{\downarrow}\Box$ with three modalities: $\Diamond_{\mathsf{u}}$ (to move one step up in the grid), $\Diamond_{\mathsf{r}}$ (to move one step to the right in the grid), and $\Diamond_{\mathsf{g}}$ (to reach all the points of the grid), interpreted by the accessibility relations U, R and G, respectively. The proof in [14] defines a hybrid formula π_T, containing the pattern $\Box{\downarrow}\Box$, that describes a tiling of $\mathbb{N} \times \mathbb{N}$ using a given finite set T of tile types. The formula π_T is the conjunction of four formulae, named α, β, γ and δ, and expressing, respectively: the existence of a "spypoint", from which the entire grid can be accessed via the relation G (α); the fact that the relations R and U are total functions (β); the "grid property" (γ); and the fact that the grid must be well-tiled (δ).

The formulae β and γ used in [14] contain the pattern $\Box{\downarrow}\Box$. Such formulae can be replaced by use of number restrictions and the satisfaction operator, using:

$$\beta = \Box_{\mathsf{g}}\Diamond_{\mathsf{u}}\top \wedge \Box_{\mathsf{g}}\Diamond_{\mathsf{r}}\top \wedge \Box_{\mathsf{g}}\Box^1_{\mathsf{u}}\bot \wedge \Box_{\mathsf{g}}\Box^1_{\mathsf{r}}\bot, \text{ and}$$
$$\gamma = \Box_{\mathsf{g}}{\downarrow}x.\Diamond_{\mathsf{u}}\Diamond_{\mathsf{r}}{\downarrow}y.x : \Diamond_{\mathsf{r}}\Diamond_{\mathsf{u}}y$$

It is worth pointing out that the universal graded modality occurs in the scope of a universal modality in the formula β (see Theorem 2 below).

The reduction of the $\mathbb{N} \times \mathbb{N}$ tiling problem to the satisfiability of π_T can be proved in the standard way (details can be found in [5]).

The $\mathbb{N} \times \mathbb{N}$ tiling problem can also be encoded in $\mathsf{HL}_m({\downarrow}, \Diamond^-, \Diamond^1) \setminus \Box{\downarrow}\Box$, by use of the same formulae α, β and δ given above, and the following ones, replacing γ:

$$\beta' = \Box_{\mathsf{g}}\Box^1_{\mathsf{u}^-}\bot \wedge \Box_{\mathsf{g}}\Box^1_{\mathsf{r}^-}\bot,$$
$$\gamma^- = \Box_{\mathsf{g}}\Box_{\mathsf{u}}\Box_{\mathsf{r}}{\downarrow}x.\Diamond_{\mathsf{r}^-}\Diamond_{\mathsf{u}^-}\Diamond_{\mathsf{r}}\Diamond_{\mathsf{u}}x$$

□

When concerned with exploring the frontiers of decidability, it might be interesting to verify whether the number of modalities used in the proof of the above theorem may be reduced or the considered fragments with less than three modalities have a decidable satisfiability problem.

The rest of this section shows how to restrict the use of graded modalities so as to obtain a decidable sublogic of $\mathsf{HL}_m(@, {\downarrow}, \mathsf{E}, \Diamond^-, \Diamond^n) \setminus \Box{\downarrow}\Box$. The expressive power of $\mathsf{HL}_m(@, {\downarrow}, \mathsf{E}, \Diamond^-, \Diamond^n)$ is actually the same as $\mathsf{HL}_m(@, {\downarrow}, \mathsf{E}, \Diamond^-)$, since graded modalities can be expressed in terms of the binder (see the proof of Theorem 2). The limitations on the use of $\Box^n_R$, in order to keep a decidable satisfiability problem, are however much stronger than those required for the other universal modalities. Moreover, occurrences of existential graded modalities have to be restricted, too.

Theorem 2. *The satisfiability problem for a formula G (in NNF) belonging to the fragment $\mathsf{HL}_m(@, {\downarrow}, \mathsf{E}, \Diamond^-, \Diamond^n) \setminus \Box{\downarrow}\Box$ is decidable provided that:*

1. *no subformula $\Box^n_R F$ of G occurs in the scope of any universal modality;*
2. *no subformula $\Diamond^n_R F$ of G is both in the scope of and contains in its scope a universal modality.*

Proof. The proof shows how to express graded modalities as abbreviations, yielding formulae that, under the additional restrictions 1 and 2, do not contain the pattern $\Box{\downarrow}\Box$.

Existential graded modalities can easily be expressed as abbreviations of formulae in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-)$: $\Diamond^n_R F$ is equivalent to the formula $(\Diamond^n_R F)^*$ defined below, where the state variables $x, y_1, \ldots, y_n$ do not occur free in F.

$$
\begin{array}{lll}
(\Diamond^0_R F)^* & \equiv_{def} & \Diamond_R F \\
(\Diamond^1_R F)^* & \equiv_{def} & \downarrow x.\Diamond_R(F \wedge \downarrow y_1.x{:}\,\Diamond_R(F \wedge \neg y_1) \\
(\Diamond^2_R F)^* & \equiv_{def} & \downarrow x.\Diamond_R(F \wedge \downarrow y_1.x{:}\,\Diamond_R(F \wedge \neg y_1 \wedge \downarrow y_2.x{:}\,\Diamond_R(F \wedge \neg y_1 \wedge \neg y_2)) \\
\ldots & & \\
(\Diamond^n_R F)^* & \equiv_{def} &
\end{array}
$$

$$
\begin{array}{l}
\quad \downarrow x.\Diamond_R(F \wedge \\
\qquad\qquad \downarrow y_1.x{:}\,\Diamond_R(F \wedge \neg y_1 \wedge \\
\qquad\qquad\qquad\qquad \downarrow y_2.x{:}\,\Diamond_R(\cdots \wedge \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow y_{n-1}.x{:}\,\Diamond_R(F \wedge \neg y_1 \wedge \cdots \wedge \neg y_{n-1} \wedge \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow y_n.x{:}\,\Diamond_R(F \wedge \neg y_1 \wedge \cdots \wedge \neg y_n)) \ldots)))
\end{array}
$$

It is easy to see that, if F belongs to $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\downarrow\Box$, then so does the formula $(\Diamond^n_R F)^*$. However, since $(\Diamond^n_R F)^*$ contains binders, if it occurs as a subformula of a formula G, in order for G to belong to the considered fragment, either no universal operator must scope over $(\Diamond^n_R F)^*$, or F must contain no universal operators.

Considering that $\Box^n_R F \equiv \neg\Diamond^n_R \neg F$, universal graded modalities can obviously be expressed in terms of the binder, too. However, the NNF of $\neg(\Diamond^n_R \neg F)^*$ contains the critical pattern $\Box\downarrow\Box$, so that resorting to the definition of $\Box^n_R$ in terms of $\Diamond^n_R$ is of no help to the aim of establishing decidability results for the hybrid language including graded modalities.

However, $\Box^n_R F$ can also be defined in a different way (here again, it is assumed that the variables $x, y_1, \ldots, y_n$ do not occur free in F):

$$
\begin{array}{lll}
(\Box^0_R F)^* & \equiv_{def} & \Box_R F \\
(\Box^1_R F)^* & \equiv_{def} & \Box_R F \vee \downarrow x.\Diamond_R(\downarrow y_1.x{:}\,\Box_R(F \vee y_1)) \\
(\Box^2_R F)^* & \equiv_{def} & \Box_R F \vee \downarrow x.\Diamond_R(\downarrow y_1.x{:}\,\Diamond_R(\downarrow y_2.x{:}\,\Box_R(F \vee y_1 \vee y_2))) \\
\ldots & & \\
(\Box^n_R F)^* & \equiv_{def} & \Box_R F \vee \\
 & & \quad \downarrow x.\Diamond_R(\downarrow y_1.x{:}\,\Diamond_R(\downarrow y_2.x{:}\,\Diamond_R(\ldots \\
 & & \qquad\qquad\qquad \downarrow y_n.x{:}\,\Box_R(F \vee y_1 \vee \cdots \vee y_n)) \ldots)))
\end{array}
$$

A reasoning showing that that $(\Box^n_R F)^*$ is equivalent to $\Box^n_R F$ can be found in [5].

Since $(\Box^n_R F)^*$ contains the pattern $\downarrow\Box$ (unless $n = 0$), $\Box^n_R F$ itself must not occur in the scope of a universal modality. This is sufficient to ensure that $(\Box^n_R F)^*$ belongs to the considered decidable fragment of HL, provided also that $\Box^n_R F$ does not contain the pattern $\Box\downarrow\Box$ (so that F does not contain the pattern $\downarrow\Box$).

$\Box$

It is worth pointing out that assuming that $\Box^n_R F$ does not occur in the scope of a universal modality would allow one to use the same "skolemization-like" transformation used in [14] (Theorem 1), and the similar one defined in [15] to reduce the consistency of cardinality restrictions to that of concept inclusion, in the presence of nominals.

4 Concluding Remarks

This work presents a satisfiability decision procedure for hybrid formulae in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box{\downarrow}\Box$. It is also proved that, although a restricted use of graded modalities can be added to the considered fragment without endangering decidability, in general, their addition to either $\mathsf{HL}_m(@, \downarrow) \setminus \Box{\downarrow}\Box$ or $\mathsf{HL}_m(\downarrow, \Diamond^-) \setminus \Box{\downarrow}\Box$ (in the simple form of functional restrictions) results in logics with an undecidable satisfiability problem.

The proof procedure has been implemented in a prover called Sibyl, which is available at `http://cialdea.dia.uniroma3.it/sibyl/`. It is written in Objective Caml and runs under the Linux operating system. Sibyl takes as input a file containing a set of assertions and a set of formulae, checks them for satisfiability and outputs the result. Optionally, a LaTeX file with the explored tableau branches can be produced. Every input formula in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box{\downarrow}\Box$ is preprocessed and translated into the fragment $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus {\downarrow}\Box$, by use of the multi-modal analogous of the satisfiability preserving translation defined in [14]. Some first experiments with the prover were carried out in order to test it for correctness. The test sets, the detailed experimental results and diagrams summarizing them, as well as some tools used for the experiments, are available at Sibyl web page. A brief summary of the results of the experiments is also reported in [4].

The core of the proof procedure is a tableau calculus where transitivity and relation inclusion assertions are treated by expansion rules which are very close to (though not exactly the same as) the analogous rules presented in [8, 9, 10, 11]. The main result of this work is proving that they can be added to a calculus dealing also with restricted occurrences of the binder, maintaining termination, beyond soundness and completeness.

Other works have addressed the issue of representing frame properties and/or relation hierarchies in tableau calculi for binder-free hybrid logic (for instance, [1, 10, 11]). The perhaps richer calculus of this kind is [10], that considers graded and global modalities, reflexivity, transitivity and role hierarchies. Converse modalities are however missing, and inverse relations are not allowed.

It is worth pointing out that the standard reduction technique dealing with transitivity and role inclusions [16] cannot be employed in the present context. In fact, that encoding makes use of role conjunction, and the latter, though expressible by use of the binder, would in general be translated into formulae falling outside the decidable fragment considered in this paper.

The scope and interest of the logic considered in this work is widened by the fact that $\mathsf{HL}_m(@, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq)$ subsumes the expressive description logic $\mathcal{SHOI}$. In particular, in the presence of the global modality, the satisfiability and subsumption problems for $\mathcal{SHOI}$ with respect to general axioms can be reduced to the satisfiability problem for $\mathsf{HL}_m(@, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq)$. This work shows that restricted uses of the binder could be added to $\mathcal{SHOI}$: binders may occur in a general concept inclusion $C \sqsubseteq D$, provided that C does not contain the pattern $\downarrow\Diamond$ and D does not contain the pattern $\downarrow\Box$. If the TBox of a knowledge base $\mathcal{K}$ satisfies this condition, the subsumption problem $\mathcal{K} \models C \sqsubseteq D$ can be reduced to the satisfiability problem for $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box\downarrow\Box$ if both C and $\neg D$ are in the fragment. The same holds for checking consistency of a concept belonging to the fragment.

The possibility of adding limited uses of the binder to description logics has been addressed, for instance, in [6, 7, 13]. In the cited works, however, the restrictions on the interplay between the binder and universal quantification is orthogonal to the one considered in the present work. Occurrences of the universal quantifier in the scope of the binder are in fact restricted so that their scope is, in turn, a negated variable. The termination proof of the system presented in this work cannot easily be extended to cover occurrences of such a pattern and whether the restriction to formulae without the pattern $\Box\downarrow\Box$ can be relaxed is an open question.

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